NONLINEAR DYNAMICS AND PSEUDO-PRODUCTION FUNCTIONS

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INTRODUCTION

The aggregate production function is a fundamental neoclassical construct. At the theoretical level, it is used in virtually every branch of economic analysis. At the empirical level, it is used to analyze the determinants of technical change and capacity utilization, and almost half a century after Solow's celebrated 1957 article, it remains *the* method of accounting for the determinants of growth. Yet the theoretical foundations of this construct are shaky, because it cannot be grounded in any plausible micro-foundations [Samuelson, 1962; 1966; 1979; Garegnani, 1970; Fisher 1971a, b; 1987; 1993; Harcourt, 1972; 1976; 1994; Solow, 1987, 25; McCombie, 2000-2001, 268; Felipe and Holz, 1999; Felipe and Adams, 2005]. It is curious that a tradition so insistent on the necessity of micro-foundations should rely so heavily on a construction that cannot be derived from micro-foundations.

Defenders claim that aggregate production functions are worth retaining because they possess important virtues, and because they appear to work at an empirical level. Paul Douglas [1976, 914, cited in McCombie and Dixon, 1991, 24] expresses this sentiment most openly: "A considerable body of independent work tends to corroborate the original Cobb-Douglas formula, but, more important, the approximate coincidence of the estimated coefficients with the actual shares received also strengthens the competitive theory of distribution and disproves the Marxian."

Robert Solow, by far the most important contributor to this tradition, takes a more nuanced position, but comes to the same conclusion: "The current state of play with respect to the estimation and use of aggregate production functions is best described as Determined Ambivalence. We all do it and we all do it with a bad conscience…One or more aggregate production functions is an essential part of every complete macroeconometric model…It seems inevitable…There seems no practical alternative… [Yet, n]obody thinks there is such a thing as a 'true' *aggregate* production function. Using an estimate of a relation that does not exist is bound to make one uncomfortable" [Solow, 1987, 15].

Despite these misgivings, Solow contends that aggregate production functions continue to be used because they appear to work: they provide "a practical way of representing the relation between the availability of inputs and the capacity to produce output" [Solow 1987, 16], while also providing a way "to reproduce the distributional

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facts" in a manner that "reinforce[s] the marginal productivity theory ... of distribution" [Solow 1987, 16-17].

It is worth emphasizing that a "good" fit[1] between aggregate output and variables such as capital, labor, and time can arise from a wide variety of function forms, ranging from ones with fixed input-output coefficients to those with smoothly variable ones. But even smoothly variable coefficients are not sufficient, since they might not be neoclassical in character. For any such good empirical fit to be read as supportive of neoclassical theory, therefore, something more is required. Two further conditions are critical. First, the smoothly varying coefficients must be part of a functional form representing a "well-behaved" neoclassical production function (Cobb-Douglas, CES, Translog, etc.). Second, the function must have estimated output elasticities matching observed wage and profit (factor) shares, thus providing support for the marginal productivity theory of distribution. As Solow once remarked, "had Douglas found labor's share to be 25 per cent and capital's 75 per cent, we should not now be talking about aggregate production functions" [McCombie 2000-2001, 269, footnote 1, quoting a remark by Solow to Fisher, cited in Fisher 1971b].

This leads us to the central issues in the debate about neoclassical aggregate production functions. Do aggregate production functions really "work" in the preceding sense? When they do appear to work, can this be taken as evidence supporting the neoclassical theory of production and distribution? And finally, can they provide reliable measures of technical change and a decomposition of the sources of growth?

To address these issues, we use two different data sets. The first set is derived from Goodwin's model of Marx's theory of persistent unemployment. The fact that it has fixed coefficient technology means that marginal products cannot even be defined, while the fact that it exhibits Harrod-Neutral technical change means that not even Samuelsonian "surrogate" marginal products can be constructed [Shaikh, 1987]. And its Marxian provenance is particularly apposite in the light of Douglas' previously cited claim that his empirically fitted function "disproves the Marxian [theory of distribution]." The second set is actual data for the U.S. Thus we have a control group whose generating process is transparent and strictly non-neoclassical, and a data set whose generating process is the object of dispute. The two data sets look very similar. In both cases, the wage shares are roughly stable, so that the Cobb-Douglas is the appropriate neoclassical production function to test. In both cases, standard fitted functions do *not* work well.

The next section explains the fundamental difficulty of distinguishing between a hypothesized neoclassical aggregate production function and a national accounting identity. Section 3 introduces our two data sets and Section 4 investigates their econometric properties. Section 5 derives "Perfect Fit" procedures that make it possible to transform a fitted production function that does not work well into one that *appears* to work perfectly. Section 6 provides a summary and conclusions.

THE SIGNIFICANCE OF THE ACCOUNTING IDENTITY

If we define Y_t, L_t, K_t, and w_t as real output, labor, capital, and the real wage, respectively, then the observed profit rate r_t = profits/capital = $(Y_t - w_t \cdot L_t)/K_t$. This yields an accounting identity that is linear in Y, K, L, and that always "adds up".

(1) $$Y_t = w_t \cdot L_t + r_t \cdot K_t.$$

A hypothesized production relation of the general form

(2) $$Y_t = F(L_t, K_t)$$

may represent many different underlying conditions, however. It may be a fixed-coefficient technology with a single technique dominating all others in wage-profit (factor-price) space, as is implicit in Harrod, Goodwin, and many others [Shaikh 1987]. It may represent a jumpy input-output relation along a wage-profit frontier with kinks at switch points from one technique to another [Michl, 1999, 196]. Or it may represent a set of smoothly varying coefficients, either because the wage-profit frontier corresponds to an infinite spectrum of fixed-coefficient methods of production [Garegani, 1970] or because it represents the aggregation of micro-level production functions [Fisher, 1971b; 1987; 1993]. In none of these cases is the functional form $Y = f(K, L)$ necessarily "well-behaved" in the traditional neoclassical sense. On the contrary, even when the coefficients are smoothly varying, one can get aggregate relations that appear to be horrendously ill-behaved [Garegnani, 1970, 430]. As Fisher [1993] has emphasized, it does not even help to begin by assuming well-behaved microeconomic production functions, because the conditions needed to produce a satisfactory aggregate relation are impossibly stringent.

But suppose that we simply posit the existence of an (approximate) aggregate production function in which factor prices equal corresponding marginal products, and in which constant returns to scale obtain (so that the factor-price-weighted sum of inputs "add up" to total output). These additional assumptions then superimpose on Equation (2) the further conditions

(3) $$\partial Y_t/\partial L_t \equiv MPL_t = w_t$$

(4) $$\partial Y_t/\partial K_t \equiv MPK_t = r_t$$

(5) $$Y_t = MPL_t \cdot L_t + MPK_t \cdot K_t$$

(from the assumption of constant returns to scale).

Equations (2)-(5) embody the standard neoclassical assumptions about aggregate production. Together, they imply that

(6) $$Y_t = w_t \cdot L_t + r_t \cdot K_t.$$

The trouble is that this relation already holds in the form of the accounting identity (Equation (1)), *quite independently* of any specification of production or distribution relations. It follows that imposing standard neoclassical assumptions about aggregate production makes *it impossible to distinguish the neoclassical argument from a mere tautology*. As Solow [1974, 121] notes, the only real function of these assumptions is to interpret the accounting identity.

But to leave it at that would imply that the most fundamental construct of neoclassical macroeconomics is a mere article of faith [Ferguson, 1971]. Solow, therefore, goes on to specify what he considers to be an adequate test of the standard neoclassical hypotheses: "When someone claims that aggregate production functions work, he means a) that they give a good fit to input-output data *without* the intervention of data deriving from factor shares; b) that the function so fitted has partial derivatives that closely mimic observed factor prices...[and c) since] technical change is always represented by a smooth function of time (or something else)...part of the test is whether the residuals are well-behaved" [Solow, 1974, 121 and footnote 1].

As already noted, the first two are required by aggregate production function theory, but the third is merely standard econometric practice, since nothing in the theory requires technical change to be a smooth function of time [Shaikh, 1980, 86-87; Felipe and Adams, 2005, 435; McCombie, 1998; McCombie, 2000-2001, 281-82]. For instance, if the pace of neutral technical change varied with the rate of growth, then the rate of technical change itself would be pro-cyclical and possibly highly variable. With that in mind, we consider whether aggregate production functions do indeed "work" in Solow's sense. But first, we need to address the issue of the data.

TWO AGGREGATE DATA SETS: ACTUAL AND CONTROL

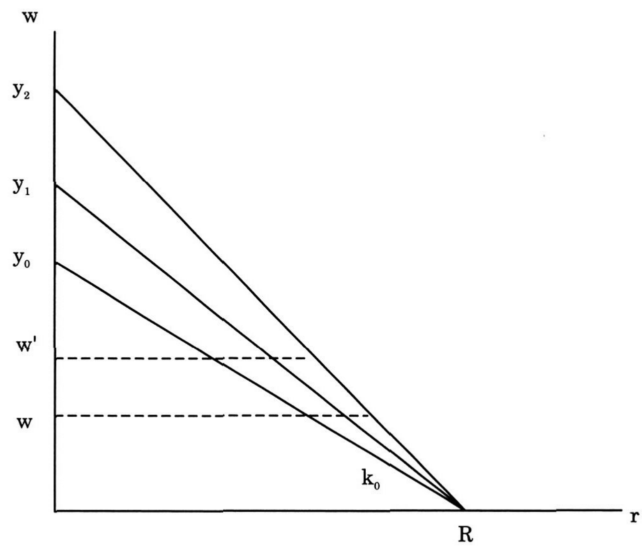
Solow tells us that aggregate production functions "work" when they fit the data well, when their coefficients yield marginal products that mimic factor shares, and when the implied pattern of technical change appears plausible. What we need to know is whether these conditions are sufficient to distinguish between neoclassical and non-neoclassical production relations. In other words, we need a control group to which we can also apply our tests.

Data set A is the control data generated from a simulation run of a slightly modified version of the Goodwin [1967] model. The original Goodwin model is, as Solow [1990, 35-36] observes, a "beautiful paper" that "does its business clearly and forcefully." Its dynamics turn on the interactions between the wage share, the rate of growth, and the employment ratio. Two changes are made here. The model is extended by allowing for a savings rate less than one (Goodwin originally assumed that all profits are saved); and Goodwin's original real-wage Phillips curve is modified by allowing for an "employer resistance" drag on real wage growth as the wage share rises (the rate of profit falls). This latter modification is made in order to produce a version of the model that is stable in the presence of stochastic shocks.[2]

There are two parts to the logic of the Goodwin model. The first has to do with the nature of the technology and its change over time. Like Harrod, Goodwin assumes that the economy is moving along its warranted path, so that output is equal to capacity. At any moment of time, a single linear fixed-coefficients technology dominates the wage rate-profit rate (factor-price) frontier, whose intercepts can be characterized by the productivity of labor and by the capacity-capital ratio. Over time, technical change is embodied in new technologies with higher capital-labor ratios that yield higher labor productivity, both of which rise at the same rate so that the capacity-capital ratio remains unchanged (this is Harrod-Neutral technical change). The assumption that coefficients are fixed at any moment of time means that marginal products cannot even be defined for any given technology. And the assumption of Harrod-Neutral technical change means that the choice of technique is invariant to the distribution of income, so that an incremental change in (say) the wage rate cannot even be associated with some corresponding change in labor productivity or in the capital-labor ratio. This excludes not only smooth "surrogate" correlations between real wages and the incremental productivity of labor [Samuelson, 1962; 1966] but also any lumpy ones [Michl 1999, 200-201]. The assumed technological structure thus excludes both actual and surrogate marginal productivity conditions. *It follows that the technological structure of this control group model is entirely distinct from that of neoclassical aggregate production function theory and associated marginal productivity rules.*

Figure 1 illustrates this aspect of the model, as taken from Shaikh [1987]. Here, the vertical axis represents the real wage and the horizontal axis the profit rate. Each technology is characterized by a linear trade-off between the wage rate and the profit rate, with limits arising from the fact that a given productivity of labor (y) is the maximum real wage, and that a given capacity-capital ratio (R) is the maximum rate of profit. The slope of each such line is the capital-labor ratio corresponding to that particular technology. The productivity of labor rises over time, but the capacity-capital ratio is constant. Thus at given real wage rates (w, w') below the existing maximum, the latest technology is dominant. Changing the real wage from w to w', for instance, will not change the chosen technology and hence will not affect labor productivity or the capital-labor ratio.

FIGURE 1
Fixed-Coefficient Technology with Harrod Neutral Technical Change



The second part of the model has to do with the dynamic interaction between the wage share and the employment ratio. Movements of the wage share $u = w/y$ are influenced by two factors: the constant rate of growth of labor productivity (α), and the rate of growth of real wages, which depends positively on the employment ratio (v) and negatively on the (squared) level of the wage share. Movements of the employment ratio, in turn, depend on three factors: the constant rate of growth of the labor force (β); the rate of growth of labor productivity (α); and the rate of growth of real output. The employment ratio and the wage rate are then linked by the fact that the wage share influences the profit rate, which influences the rate of growth of capital and hence the growth rate of real output.[3] Since the model is stable, in the absence of shocks the growth rate of output converges to the natural growth rate $(\alpha + \beta)$, the wage share converges to some constant level u^*, and the employment share to some constant level v^* that is less than one (signifying a persistent rate of unemployment). This modified Goodwin model is summarized in Appendix A.

The other data series used in this paper (data set B) is actual data from the U.S. Bureau of Economic Analysis (BEA) National Income and Product Accounts (NIPA),

FIGURE 2
Output (*Y*) and Capital (*K*)

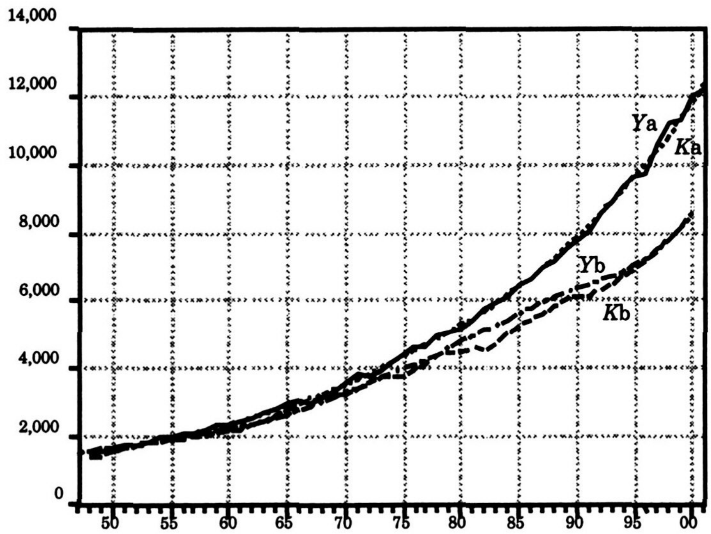
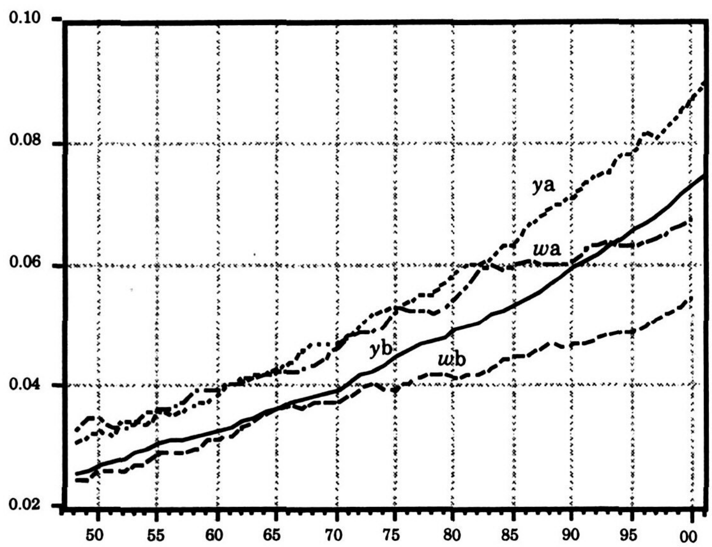


FIGURE 3
Real Wages (*w*) and Labor Productivity (*y*)



and from corresponding wealth stocks. This gives us two data sets, the Marx-Goodwin simulation data set (A) and the actual U.S. data set (B), both of which satisfy the accounting identity of Equation (1). Figure 2 displays paths of output (Y) and capital (K), Figure 3 real wages (w) and productivity (y), Figure 4 the profit rate (r), and Figure 5 the wage share (u) and the employment ratio (v).

FIGURE 4
Profit Rate (r)

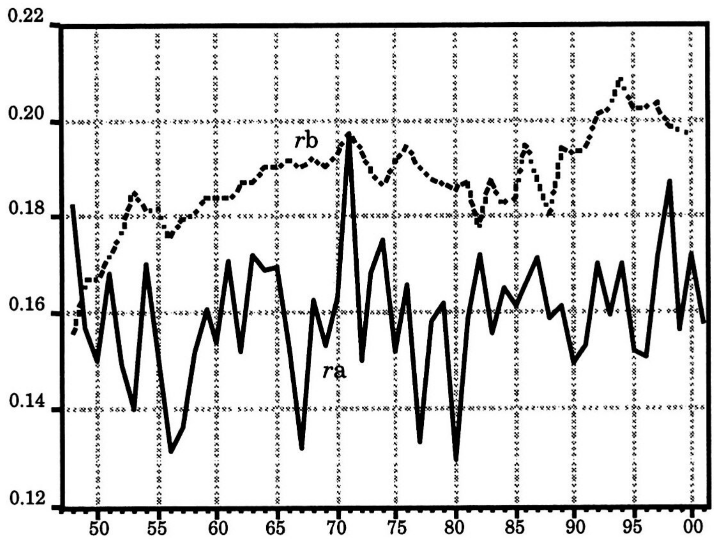
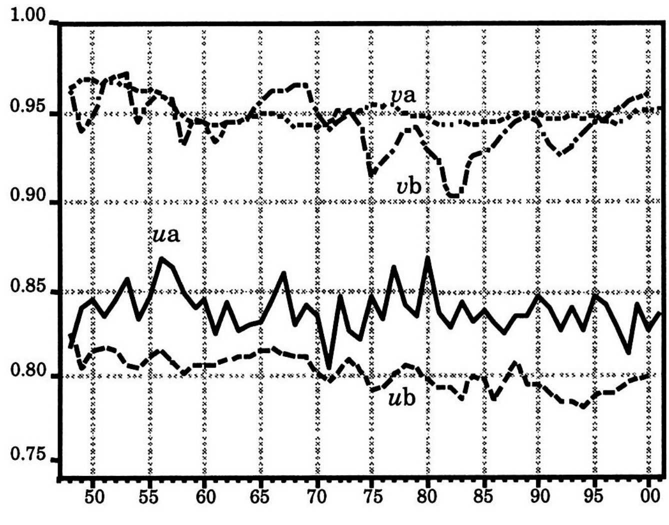


FIGURE 5
Wage Share (u) and Employment Ratio (v)



DO AGGREGATE PRODUCTION FUNCTIONS "WORK" AT AN EMPIRICAL LEVEL?

Figure 5 shows that the wage shares in data sets A and B are roughly stable, with means of $u_a \approx 0.84$ and $u_b \approx 0.81$, respectively. This means that a Cobb-Douglas function is an appropriate starting point to test neoclassical aggregate production function theory (although it is theoretically inappropriate for data set A). We work with the standard form in which technical change is assumed to be neutral ($Y_t = A_t{\cdot}L_t^{b}{\cdot}K_t^{c}$), coefficients b and c represent the putative factor shares, and their sum represents the degree of returns to scale. If we wish to impose the further restriction of constant returns to scale ($b + c = 1$), we can divide by labor to get the per employee form ($y_t = A_t{\cdot}k_t^{c}$), in which the coefficient c once again represents the profit share implied by the marginal productivity theory of distribution.

For the purpose of empirical estimation, we express the regression forms in both levels and growth rates. As is standard, the technical change parameter is expressed as a log-linear function of time, since quadratic and cubic time terms did not change the basic results of the regressions. This gives us four regressions altogether and two data sets for each. All regressions are OLS, as is customary in this literature, and the error term is represented by ε. Of particular interest are the relations between estimated coefficients and the corresponding actual labor and capital shares. Table 1 reports the results of runs of each equation on both data sets.

(7) $$\log Y_t = a_0 + a_1{\cdot}t + b{\cdot}\log L_t + c{\cdot}\log K_t + \varepsilon$$

(8) $$\Delta\log Y_t = a_0 + a_1{\cdot}t + b{\cdot}\Delta\log L_t + c{\cdot}\Delta\log K_t + \varepsilon$$

(9) $$\log y_t = a_0 + a_1{\cdot}t + c{\cdot}\log k_t + \varepsilon$$

(10) $$\Delta\log y_t = a_0 + a_1{\cdot}t + c{\cdot}\Delta\log k_t + \varepsilon$$

The first pair of regression forms do not assume constant returns to scale, so the sum of the labor and capital coefficients are not restricted in advance. When run in levels, the overall fit is excellent, and the labor coefficient is significant and large for both data sets. In set A the time trend and capital coefficients are not significant but the overall D.W. statistic is quite good (2.117), while in set B the time trend and capital coefficients are significant but the D.W. is not good (0.219). In neither set are the implied shares close to the actual, and constant returns to scale never obtains. When run in rates of growth, the overall fits are quite good for both sets of data, the time trends are significant, the labor coefficients are close to one and highly significant, and the D.W.'s are good. But in both cases the capital coefficient is negative, so that implied shares are very different from actual ones.

The second pair of regressions restricts the coefficients to sum to one (that is, they assume constant returns to scale), so the relevant variables are output and capital per employee. In levels, the overall fit is once again excellent, and the constants and time trend are highly significant. In set A, however, the coefficient of the capital-labor ratio is small and not statistically significant, while the overall D.W. is quite good; in set B, the coefficient of the capital-labor ratio is relatively large but the D.W.

quite low. Once again, the estimated capital coefficient is not even close to the actual profit share in either set. Finally, when run in growth rates, only the constant is significant, implying significant positive rates of neutral technical change, while all other results are generally quite bad. On the whole, despite the fact that the wage shares are roughly stationary in both data sets, none of the fitted forms of the Cobb-Douglas aggregate production function work well on either the simulated data (set A) or the actual data (set B).

TABLE 1
Cobb-Douglas Production Functions Fitted to Actual and Simulated Aggregate Data (OLS)
(1948-2000 for Levels and 1949-2000 for First Differences)

Dependent Variable	$\log Y_t$ Data A	$\log Y_t$ Data B	$\Delta\log Y_t$ Data A	$\Delta\log Y_t$ Data B	$\log y_t$ Data A	$\log y_t$ Data B	$\Delta\log y_t$ Data A	$\Delta\log y_t$ Data B
Constant	–4.628*	–0.279	0.109*	0.035*	–3.453*	–2.109*	0.019*	0.022*
	(1.722)	(1.900)	(0.026)	(0.009)	(0.358)	(0.462)	(0.005)	(0.004)
Time	0.0134	0.0133*	0.00013	–0.00025	0.020*	0.009*	3.80E-05	–0.0002
	(0.009)	(0.005)	(0.00013)	(0.00013)	(0.002)	(0.002)	(0.00014)	(0.0001)
$\log L_t$	0.989*	0.471*	—	—	—	—	—	—
	(0.103)	(0.191)						
$\log K_t$	0.170	0.341*	—	—	—	—	—	—
	(0.240)	(0.145)						
$\Delta\log L_t$	—	—	0.998*	0.972*	—	—	—	—
			(0.096)	(0.096)				
$\Delta\log K_t$	—	—	–2.315*	–0.351	—	—	—	—
			(0.659)	(0.251)				
$\text{Log}k_t$	—	—	—	—	0.019	0.395*	—	—
					(0.102)	(0.135)		
$\Delta\log k_t$	—	—	—	—	—	—	–0.024	0.043
							(0.106)	(0.098)
Adj. R^2	0.9997	0.9952	0.6916	0.6912	0.9988	0.9760	–0.0382	0.0276
D.W.	2.103	0.219	2.344	2.076	2.026	0.185	2.943	2.041
Implied Wage Share	0.989	0.471	0.998	0.972	0.981	0.605	1.0024	0.957
Actual Wage Share	0.840	0.810	0.840	0.810	0.840	0.810	0.840	0.810
Implied Profit Share	0.176	0.341	–2.315	–0.351	0.019	0.395	–0.0024	0.043
Actual Profit Share	0.160	0.190	0.160	0.190	0.160	0.190	0.160	0.190
Implied Returns to Scale	1.165	0.812	–1.317	0.621	—	—	—	—

Notes: Standard errors statistics are listed below estimated coefficients. Starred coefficients imply significance at 5 percent or better.

Are these results typical? Douglas seemed to think not [1976, 914]. Samuelson [1979, 924] points out, however, that Douglas' own regressions did not include a term for technical change, and Felipe and Adams [2005, 429-30] show that when a term for neutral technical change is introduced, Douglas' original data set yields a "coefficient of the index of capital which is negative and insignificant."

Solow initially emphasized the importance of the similarity between Douglas' estimated parameters and actual factor shares [Fisher, 1971b, in McCombie, 2000-2001, 269]. He repeated this sentiment in his first response to Shaikh [1974]. Having found that the OLS regression of log y on log k in Shaikh's constructed data yields a result in which the "point estimate of log k is negative" and not statistically significant, Solow says that if "this were the typical outcome with real data, we would not now be having this discussion" [1974, 121]. And yet it turns out that the very same test on his own data would have given similar results. McCombie [2000-2001, 281-283] revisits Solow's original data and comments that "it is surprising that Solow did not seek to [similarly] 'test' the Cobb-Douglas function using his own data." For if he had, then he would have found that when run in levels "the coefficient of capital term is not statistically [different] from zero,"[4] and when run in ratios "the coefficient of the capital-labor term is negative, but statistically insignificant." McCombie goes on to remark that we "can only speculate whether Solow's [1957] paper would have had such a dramatic impact if these regressions had also been reported."

It turns out that such results are indeed quite typical down to the finding of negative capital coefficients [Sylos-Labini, 1995; Felipe and Adams, 2005, 429-30]. Nonetheless, aggregate production functions do appear to work on occasion. Can we then say that, at least in these cases, a good fit provides some evidence on the underlying production structure and on the marginal productivity theory of distribution?

HOW TO MAKE AGGREGATE PRODUCTION FUNCTIONS ALWAYS "WORK PERFECTLY" (EVEN WHEN COMPLETELY INAPPROPRIATE)

The purpose of this section is to show that *one can always construct an infinite number of empirically fitted aggregate productions that work "perfectly."* The secret lies in the specification of the function of time representing technical change. In the present case we are concerned with data with roughly stable wage shares, to which we fit Cobb-Douglas type regressions in either growth rates or in log levels. We illustrate the procedure with regressions involving rates of change (for example, the second and fourth types in Table 1), with some general function of time F(*t*) in place of the previously assumed time variable (*t*).

(11) $$\Delta \log Y_t = a_0 + a_1 \cdot F(t) + b \cdot \Delta \log L_t + c \cdot \Delta \log K_t + \varepsilon$$

(12) $$\Delta \log y_t = a_0 + a_1 \cdot F(t) + c \cdot \Delta \log k_t + \varepsilon$$

The forms of regression Equations (11) and (12) derive from the assumption that *Y*, *K*, and *L* are bound together by a *hypothesized* Cobb-Douglas production function with neutral technical change. These very same variables are also bound together by the *actual* accounting identity $Y = w \cdot L + r \cdot K$ and its per employee form $y = w + r \cdot k$. Differencing these identities and leaving out cross-products of first differences, we derive the two rate-of-change forms,[5] in which the Solow Residual SR(*t*) is the share-weighted average rates of change of the real wage and profit rate.

(13) $$\Delta(\log Y_t) \equiv SR(t) + u_{t-1} \cdot \Delta(\log L_t) + (1 - u_{t-1}) \cdot \Delta(\log K_t)$$

(14) $$\Delta(\log y_t) \equiv \mathrm{SR}(t) + (1 - u_{t-1}) \cdot \Delta(\log k_t)$$

(15) $$\mathrm{SR}(t) \equiv u_{t-1} \cdot \Delta \log w + (1 - u_{t-1}) \cdot \Delta \log r$$

Now, if the wage share (u) is stable, so that $u(t) \cong u \cong$ constant, we can create an infinite number of time functions F(t) that will always make fitted production functions work "perfectly" in the sense of Solow: that is, make them yield perfect econometric fits with partial derivatives that closely approximate observed factor prices.

Note that with $u \cong$ constant, the accounting identities in Equations (13) and (14) look just like Cobb-Douglas production functions with a rate of neutral technical change SR(t). Therefore, *if we were to define the rate of technical change as F(t) = SR(t), the regression Equations (11) and (12) will always "pick up" the corresponding identity Equations (13) and (14).* In other words, this particular specification of technical change in the regression equations will always produce a perfect neoclassical fit—regardless of the underlying data generation process [McCombie and Dixon 1991, 27]. Comparing the two sets of equations makes it clear that we will find $a_0 = 0$ and $a_1 = 1$.

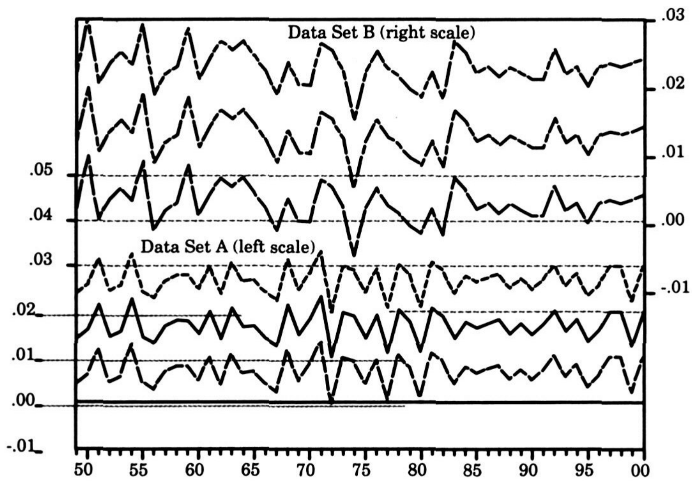
This technique embodies Solow's own original measure of technical change, which itself fluctuates substantially over time [Solow, 1957; McCombie, 2000-2001, 281-282]. As noted earlier, nothing in neoclassical theory precludes complex paths for technical change. If it is desired that technical change be represented by some smooth measure, however, this is easily accommodated. Once we recognize that setting F(t) = SR(t) will always give a perfect fit, then it is evident that making F(t) into a one-to-one function of SR(t) will also work just as well, since in both cases the two variables are perfectly correlated. One simple way to accomplish this is to define F(t) as an affine transform of SR(t) with a damping coefficient (h). Let σ= mean of SR(t). Then for any parameters $\alpha > 0$, $0 < h < 1$,

(16) $$\mathrm{F}(t) \equiv \alpha + h \cdot (\mathrm{SR}(t) - \sigma).$$

Since SR(t) is generally stationary, F(t) will be stationary also. The two series will generally have different means unless $\alpha = \sigma$ Given that F(t) represents the rate of technical change in Equations (11) and (12), its summation will represent the index of the level of technology at any moment of time. This technology index will be smoother the smaller the damping coefficient h, and will be steeper the higher the parameter α. When $h = 1$, the resulting technical level function will not generally be smooth. But by reducing h sufficiently, one can make the technology index as smooth as desired. This Perfect Fit Theorem is proven in Appendix B.

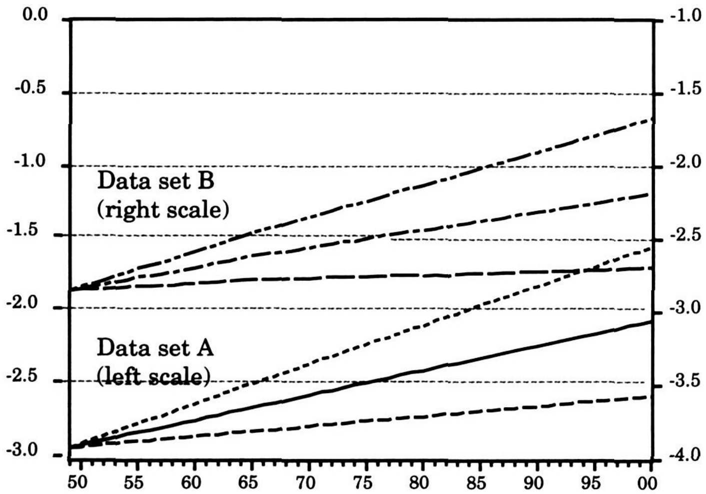
One consequence of this theorem is that there are as many perfect fits as there are values of α and h, *each of which will give a different picture of technical change.* And yet, each will be perfectly correct. For each data set Figure 6 illustrates three such specifications of the rate of technical change, F(t), which is normalized to equal the initial value of $\log y - (1 - u) \cdot \log k$. In all cases, $h = 0.2$, and for each data set the middle curve is for $\alpha = \sigma$, while the higher and lower curves are derived from $\alpha = \sigma +$ 0.01 and $\alpha = \sigma - 0.01$, respectively. Figure 7 depicts the corresponding indexes of the level of technology derived through the summation of each F(t). Table 2 illustrates the "perfect" regressions arising from $h = 0.6$, 0.2 for each data set and $\alpha = \sigma$ in all cases.

FIGURE 6
Perfect Fit Technical Change Functions F(t)



h = 0.2, and alpha is created by adding 0.01, 0.0, -0.01, respectively, to the mean of SR(t) of the particular data set. The bottom three lines are in set A, top three in set B.

FIGURE 7
Technical Level Functions (log scales)



h = 0.2, and alpha is created by adding 0.01, 0.0, -0.01, respectively, to the mean or SR(t) of the particular data set.

TABLE 2
Cobb-Douglas Production Function "Perfect Fits"
for Simulated and Actual Data in Rates-of-Change Regressions (OLS), 1949-2000

Dependent Variable	$\Delta\log Y_t$				$\Delta\log y_t$			
	Data A		Data B		Data A		Data B	
	$h = 0.6$	$h = 0.2$	$h = 0.6$	$h = 0.2$	$h = 0.6$	$h = 0.2$	$h = 0.6$	$h = 0.2$
Constant	–0.0134	–0.070	–0.008	–0.052	–0.012	–0.068	–0.009	–0.053
	(0.0005)	(0.006)	(0.0001)	(0.0002)	(9.17E–05)	(0.0002)	(5.20E–05)	(0.0001)
F(t)	1.692	5.075	1.663	4.989	1.685	5.056	1.664	4.992
	(0.004)	(0.012)	(0.003)	(0.008)	(0.004)	(0.012)	(0.003)	(0.009)
$\Delta\log L_t$	0.841	0.841	0.807	0.807	—	—	—	—
	(0.002)	(0.002)	(0.001)	(0.001)				
$\Delta\log K_t$	0.202	0.202	0.187	0.188	—	—	—	—
	(0.012)	(0.012)	(0.003)	(0.003)				
$\Delta\log k_t$	—	—	—	—	0.158	0.158	0.193	0.193
					(0.002)	(0.002)	(0.001)	(0.001)
Adj. R^2	0.9999	0.9999	0.9999	0.9999	0.9997	0.9997	0.9998	0.9998
D.W.	2.432	2.432	1.550	1.550	1.915	1.915	1.509	1.509
Implied Wage Share	0.841	0.841	0.807	0.807	0.842	0.842	0.807	0.807
Actual Wage Share	0.840	0.840	0.810	0.810	0.840	0.840	0.810	0.810
Implied Profit Share	0.202	0.202	0.188	0.188	0.158	0.158	0.193	0.193
Actual Profit Share	0.160	0.160	0.190	0.190	0.160	0.160	0.190	0.190

Note: $h = 0.2, 0.6$, $\alpha = \sigma$ throughout. For data A, $\sigma = 0.167$, and for data B, $\sigma = 0.131$. In each regression, the theoretically predicted coefficients are: constant = $(\sigma - \alpha/h)$, coefficient of F(t) = $1/h$, coefficient of $\Delta\log L_t$ = the wage share (u), and coefficients of $\Delta\log K_t$ and $\Delta\log k_t$ = the profit share $(1 - u)$.

In both data sets, all values of α and h produce close to "perfect" fits for a Cobb-Douglas production function satisfying marginal productivity rules and even exhibiting smooth technical change. *And therein lies the rub*, for we already know that data set A is generated from a Goodwin-type model with a fixed-coefficient technology undergoing Harrod-Neutral technical change. Moreover, the stability of the long-run wage share in this model derives from the classical feedback among persistent unemployment, real wages, and the rate of growth. Neither actual nor surrogate marginal products, nor any theory of wages linked to them, can even be defined within this framework.

The Perfect Fit Theorem demonstrates that there exists a wide range of smooth technical change functions of the form F(t) = f(SR(t)) that will make standard regressions work perfectly. It follows that the regressions will work almost as well if F(t) is some good approximation of SR(t), say through the use of a Fourier series [Shaikh, 1980, 86; Felipe and Adams, 2005, 435; McCombie, 1998, 167-168].

In the preceding cases, F(t) is smooth because it is in some sense a good approximation of the non-smooth Solow Residual SR(t). But we could produce the same result by redefining variables in such a way that SR(t) itself becomes smooth. Since the latter is the share-weighted average of the rates of growth of wage and profit rates (Equation (15)), any data adjustments that smooth w, r will also end up smoothing SR(t).

Such an outcome can arise simply from an attempt to adjust for cyclical fluctuations. Suppose we consider actual output (Y) to be a function of *utilized* inputs (L^*, K^*), that is, "factor services" [McCombie, 1998, 159, 167-168; 2000-2001, 285-288]. One simple way to do this is to define factor services as $L^* = z_L \cdot L$, $K^* = z_K \cdot K$, where the factor utilization rates (z_L, z_K) are themselves the ratios of actual factor productivities (y, R) to trend productivities (y^*, R^*). In log terms, this gives us

(17) $$\log z_L = \log y - \log y^*$$

(18) $$\log z_K = \log R - \log R^*$$

(19) $$\log L^* = \log L + \log z_L$$

(20) $$\log K^* = \log K + \log z_K$$

Actual output (Y) continues to be the sum of wages and profits from the accounting identity. The factor share will therefore not be affected by any transformation of variables. But the wage share is $u = w/y$ and the profit share is $(1 - u) = r/R$, so replacing actual productivities (y, R) with smooth trend productivities (y^*, R^*) will result in new, equally smooth wage and profit rates (w^*, r^*). This means that the new Solow Residual SR*(t) will also be smooth. We can even make the Solow Residual into a simple linear function of time, as is commonly assumed in production function regressions (for example, Equations (7)-(10)).

A simple illustration will suffice. If factor shares are constant, the rates of change of w^*, r^* will be exactly those of y^*, R^*, respectively. Suppose the productivity trends ($\log y^*$, $\log R^*$) are estimated as quadratic functions of time with the coefficients shown below, and we define $\beta_0 = [u \cdot m_1 + (1-u) \cdot n_1]$ and $\beta_1 = [u \cdot m_2 + (1-u) \cdot n_2]$. Then we have

(21) $$\log y_t^* = m_0 + m_1 \cdot t + m_2 \cdot t^2$$

(22) $$\log R_t^* = n_0 + n_1 \cdot t + n_2 \cdot t^2$$

(23) $$\mathrm{SR}^*(t) \equiv u \cdot \Delta \log w_t^* + (1-u) \cdot \Delta \log r_t^* = \beta_0 + \beta_1 \cdot t$$

With SR*(t) reduced to a standard linear time trend, and with factor shares roughly constant, the accounting identities are indistinguishable from the corresponding standard production function regressions (for example, Equations (13) and (14) will look just like Equations (8) and (10)). A perfectly reasonable procedure for adjusting for cyclical variations can therefore end up leading to a perfect fit—of the pseudo production function.[6]

More formally, we can always replace the term SR(t) in the accounting identity Equations (13) and (14) with some time trend f(t) and partition out the residual [SR(t) – f(t)] to labor and/or capital as "utilization" adjustments $\Delta \log z_L$, $\Delta \log z_K$, respectively. This would give new measures of utilized labor and capital ($L^* = z_L \cdot L$, $K^* = z_K \cdot K$), a new accounting identity residual SR*(t) = f(t), and a new accounting identity equation that is structurally identical to the standard production function regression. Not surprisingly, the standard regressions are then likely to pick up the pseudo-production function.[7]

This brings us to a counter-argument advanced by Solow [1987].[8] He proposes that we consider a physical production process in a single factory. Because he knows that one cannot directly observe the aggregate production process, he also excludes this possibility at the factory level. Solow then claims that, given recorded inputs and outputs at the factory level, one should be able to deduce the true microeconomic production function by econometric means alone. As he puts it, "it is simply not credible that constancy of relative shares—or anything else—can prevent us from tracing out the production function" [Solow, 1987, 19-20].

The problem is *that we do not know which particular econometric regression corresponds to the form of the true production function*. If we were allowed to examine the operations of the factory, then we could directly ascertain the underlying production process. But if we cannot do this, we can only test a variety of regression forms that we have picked on some *a priori* grounds. This is precisely where theory, and faith, enters into the story.

Consider data set A, whose non-neoclassical underlying production process is characterized by fixed-coefficients production and Harrod-Neutral technical change (so that R is roughly constant). Nothing prevents us from considering the production data in this model to be a scaled-up version of a "representative" factory. So we have before us a direct test of Solow's hypothesis. Since $R_t = y_t/k_t = Y_t/K_t$, and is roughly constant, where $y_t = Y_t/L_t$ and $k_t = K_t/L_t$, the forms of the true production function are

$$\log Y = \log R + \log K \quad (24)$$

$$\Delta \log Y = \Delta \log K \quad (25)$$

$$\log y = \log R + \log k \quad (26)$$

$$\Delta \log y = \Delta \log k \quad (27)$$

The regressions based on the preceding true forms give absolutely perfect results in every case. Thus Solow is right to say that we *can* pick up the true form. But this is only because we know it in advance. Solow's econometrician does not have this information. Being neoclassical, he or she will therefore turn to the standard regressions of Equations (7)-(10). Yet the results for data set A in Table 1 show that *not one of these comes close to identifying the true production function*. For example, in the true-form regressions corresponding to Equations (26)-(29), the estimated coefficients for K, ΔK, k, and Δk all equal 1, as they should. However, although all the standard-form regressions of Table 1, Data A, have good-to-excellent econometric properties, the corresponding capital coefficients are 0.170, –2.315, 0.019, and –0.024, respectively. Worse yet, only the second coefficient, which is highly negative, is statistically significant.

Faced with such results, how does one proceed? This is where aggregation comes in. If we consider a factory, then the answer is clear: go in and see how things work. At the aggregate level, however, we have no such option, so we turn to the theory. But once we recognize that the theory does not provide much support for the notion of aggregate production functions, we either turn away from this concept or turn back to the data to see if it is possible to improve the results. Here, as we have seen, there exist a variety of adjustments that will make matters eventually appear to come out right. Yet the resulting empirical strength of aggregate production functions and marginal productivity theory would be an illusion. In each case, the regression would be actually picking up the pseudo-production function implicit in the accounting identity, rather than the true production function.

The lesson should be clear. We know that aggregate production functions cannot be derived from micro-foundations, and we know that they generally do not work well at an empirical level. But when they do happen to work empirically, it is because the terms used to proxy the rate of technical change and/or to adjust for fluctuations in factor utilization happen to approximate the associated accounting identity residual SR(t).[9]

SUMMARY AND CONCLUSIONS

Aggregate production functions are still widely used four decades after it was conceded that they could not be grounded in any plausible micro-foundations. Their presence is generally justified on the ground that they appear to work empirically, by which it is meant that they yield a good econometric fit and have partial derivatives closely approximating factor prices. But fitted aggregate production functions do *not* generally work well in this sense, because estimated partial derivatives differ considerably from factor prices, and often even yield negative capital coefficients.

Even so, aggregate production functions do occasionally work. This paper shows that aggregate production functions can always be made to work on any data that exhibits roughly constant wage shares, even when the underlying technology is non-neoclassical. But in so doing, they always pick up the accounting identity that underlies the data. This is demonstrated on both actual U.S. data and a control data set derived from a fixed coefficient model with Harrod-neutral technical change and a persistent rate of unemployment. In the latter case, there are no marginal products. Yet one can always fit an aggregate production function that yields an excellent fit, estimated coefficients equal to factor shares, smooth technical change, and good residuals. It is proved, moreover, that one can generate an infinite number of such fits, each of which gives a different reading of the rate of technical change. It follows that even when aggregate production functions appear to work at an empirical level, they provide no support for the neoclassical theory of aggregate production and distribution. On the contrary, the best of fits can utterly misrepresent the true underlying mechanisms of production, distribution, technical change, and growth.[10]

APPENDIX A

The modified Goodwin model used in this paper was summarized by the following nonlinear system of equations. The parameter values used to generate the data are listed below the equations. Three sets of random shocks, $e_1 = e_2 = 0.001(\eta)$, and $e_3 = 0.03(\eta)$, were incorporated in the model as shown below, where η was generated from (pseudo) random draws from a normal distribution with zero mean and unit variance (this is the variable "nrnd" in Eviews 4). To mimic the actual fluctuations in output-capital ratio, the shock e_3 was multiplicatively applied to R itself.

(28) $$u_t = w_t/y_t$$
[u = wage share = real wage/labor productivity]

(29) $$v_t = Y_t/(y_t \cdot N_t)$$
[v = employment ratio = output/(labor productivity·labor force)]

(30) $$\log y_t = \log y_{t-1} + \alpha + e_1$$
[constant rate of growth of labor productivity = α]

(31) $$\log N_t = \log N_{t-1} + \beta$$
[constant rate of growth of the labor force = β]

(32) $$\log w_t = \log w_{t-1} - \gamma + \rho \cdot v_{t-1} - \rho_1 \cdot (u_{t-1})^2 + e_2$$
[real wage growth function]

(33) $$\log Y_t = \log Y_{t-1} + s \cdot (1 - u_{t-1}) \cdot R \cdot (1 + e_3)$$
[output growth rate = savings rate·profit rate]

$$\alpha = 0.02,\ \beta = 0.02,\ \gamma = 0.10,\ \rho = 0.335,\ \rho_1 = 0.28,\ s = 0.25,\ R = 1.$$

APPENDIX B: THE PERFECT FIT THEOREM

With a stable wage share, for any $\alpha > 0$, and $0 < h < 1$, a sufficiently small h will yield an F(t) such that there will be a "perfect" (or near-perfect) fit for a Cobb-Douglas production function with smooth technical change and partial derivatives that mimic factor prices.

Proof: Solve for SR(t)= $(s - \alpha/h) + (1/h)\cdot\mathrm{F}(t)$ from Equation (16) and substitute this into the accounting identity Equations (13) and (14), and noting that the wage share (u) is constant, we get

(34) $$\Delta(\log Y_t) \equiv (\sigma - \alpha/h) + (1/h)\cdot\mathrm{F}(t) + u\cdot\Delta(\log L_t) + (1-u)\cdot\Delta(\log K_t)$$

(35) $$\Delta(\log y_t) \equiv (\sigma - \alpha/h) + (1/h)\cdot\mathrm{F}(t) + (1-u)\cdot\Delta(\log k_t)$$

Comparing these to the standard regression Equations (11) and (12) makes it clear that the econometric fit will be more or less perfect, and that the estimated regression coefficients will be $a_0 = (\sigma - \alpha/h)$, $a_1 = 1/h$, $b = u$, $c = 1 - u$. The last two parameters are particularly important, since they imply that *the estimated labor and capital coefficients equal the corresponding factor shares, as hypothesized in marginal productivity theory*. The more stable the wage share, the more "perfect" will be the fitted Cobb-Douglas function. The smaller the chosen value of the parameter h, the smoother will be the apparent level of technical change.

NOTES

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1. A good fit also requires that the residuals are well-behaved [Solow, 1974, 121, footnote 1].
2. I thank Duncan Foley for suggesting this modification.
3. Beginning from the short run equilibrium condition that investment equals savings ($I = S$), and assuming that savings are proportional to profits (P) because workers do not save, we have $I = s \cdot P$. Dividing by the capital stock yields $I/K \equiv K'/K = P/K \equiv r$, where K'/K stands for the rate of growth of capital. But the profit rate $r = P/K = (P/Y) \cdot (Y/K)$ can be further decomposed by noting that the profit share $P/Y = (Y - w \cdot L)/Y = 1 - w/y = 1 - u$, where $y = Y/L$ = labor productivity, and $u = w/y$ = the wage share. Along the warranted path, output = capacity, and in the presence of Harrod-Neutral technical change, the capacity-capital ratio $R = Y/K$ = constant. Thus the rate of growth of output (Y'/Y) = the rate of growth of capacity = the rate of growth of capital $(K'/K) = s \cdot r = s \cdot R \cdot (1 - u)$.
4. McCombie's [2000-2001, 282] text actually says "not statistically significant from zero," but the meaning is clear from the context.
5. We could just as well have derived expressions in levels. Given the definition of the wage share $u = w/y$, the per-unit-labor accounting identity $y = w + r \cdot k$ implies that the profit share is $1 - u = r \cdot k/y$. Thus, $\log w = \log u + \log y$, and $\log r = \log(1 - u) + \log y - \log k$. Multiplying the first expression by u, and the second by $(1 - u)$, adding the two, and reordering terms gives us the accounting identity expression (with time made explicit) $\log y(t) = \mathrm{b}(t) + (1 - u(t)) \cdot \log k(t)$, where $\mathrm{b}(t) \equiv -[u(t) \cdot \log u(t) + (1 - u(t)) \cdot \log(1 - u(t))] + [u(t) \cdot \log w(t) + (1 - u(t)) \cdot \log r(t)]$. Adding $\log L_t$ would then give an equivalent expression in logs of Y_t, L_t, K_t. Now, if the wage share happens to be roughly constant ($u(t) \cong u$), then the accounting identity expressions "look" just like constant returns to scale Cobb-Douglas production functions with a labor coefficient b = u, capital coefficient c = $1 - u$, and some (not necessarily smooth) time function $\mathrm{b}(t)$ representing neutral technical change.
6. When the wage share is exactly constant, then smoothing y, R is exactly equivalent to smoothing w, r, and one could derive the utilization adjustments from either. Since shares fluctuate considerably in both data sets, smoothing w, r directly is much more effective than smoothing SR(t).
7. McCombie [2000-2001, 285-288] follows just such a procedure. He takes the mean of SR(t) as its smoothed value, assigns the residual to capital as a capacity utilization adjustments, and shows that this generates an excellent fit for a standard Cobb-Douglas. As he notes, this is because "we are again merely estimating the identity."
8. Solow also advances the claim that my accounting identity argument amounts to the discovery "that any production function can be written as the product of a Cobb-Douglas and something else. The something else is the production function divided by a Cobb-Douglas" [Solow, 1987]. But it should be clear from my text that the accounting identity is completely independent from any assumptions concerning the existence of an aggregate production function, Cobb-Douglas or otherwise.
9. The basic arguments can be extended to production functions other than the Cobb-Douglas. In effect, should factor shares be changing over time, the problem becomes one of accommodating the variations in both the residual SR(t) and also those in the changing wage share. In the end, it is still the accounting identity that drives it all [McCombie and Dixon].
10. It should be mentioned that it is not necessary to assume an aggregate production function to measure technical change. If we were instead to assume fixed-proportion methods for each commodity, then we could characterize technical change by its effects on the (normal capacity) rate of profit at any given wage [Sraffa, 1960; Okishio, 1961; Samuelson, 1962]. From the accounting identity written in the form $r_t = (y_t - w_t)/k_t$, holding the real wage constant and dropping cross-products of first differences, we get $\Delta r_t/r_{t-1} = \Delta y_t/(y_{t-1} - w_{t-1}) - \Delta k_t/k_{t-1} = (\Delta y_t/y_{t-1}) \cdot (1/(1 - u_{t-1})) - \Delta k_t/k_{t-1} = (\Delta y_t/y_{t-1})) \cdot (u_{t-1}/(1 - u_{t-1}) - (\Delta R_t/R_{t-1})$, since $R = Y/K = y/k$ implies that $\Delta k_t/k_{t-1} = \Delta y_t/y_{t-1} - (\Delta R_t/R_{t-1})$. Thus the Leontief-Sraffa-Samuelson-Okishio (LSSO) "choice of technique" measure of aggregate technical change is $\mathrm{t}(t) \equiv [\Delta\log(r_t)]_{\Delta w=0} = \Delta\log(y_t) \cdot (1/(1 - u_{t-1})) - \Delta\log(k_t) = \Delta\log(y_t) \cdot (u_{t-1}/(1 - u_{t-1}) - \Delta\log(R_t)$. By contrast, using $R = Y/K = y/k$, the Solow measure in Equation (15) can be written as

$SR(t) \equiv \Delta \log y_t - (1 - u_{t-1}) \cdot \Delta \log k_t = u_{t-1} \cdot \Delta \log y_t + (1 - u_{t-1}) \cdot \Delta \log R_t$. Generally speaking, $t(t)$ is far smaller than SR(t). This would suggest a different reading of growth accounts.

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